DISSERTATIO ACADEMICA

THEORIAM ÆQUATIONUM FUNCTIONALIUM DUARUM VARIABILIUM EJUSQUE IN DOCTRINA SERIERUM USUM EXHIBENS;

QUAM

CONSENSU AMPLISS. FACULTATIS PHILOSOPH.
AD IMPERIALEM ACAD. ABOËNSEM,

PRÆSIDE

Mag. NATH. G. AF SCHULTÉN,
Mathematatum Professore Publ. & Ord.,
Acad. Imperialis Scientiarum Petropolitanæ
Socio Corresp.,

PRO GRADU PHILOSOPHICO

P. P.

CAROLUS HENRICUS AHLQVIST,
Wiburgensis.

In Audit. Jurid. die XXIII Maji MDCCCXXVII.
horis a. m. solitis.

P. III.

ABOÆ, Typis FRENCKELLIANIS.

b_1	b_3	b_5	b_7	b_9	b_{11}	b_{13}	b_{15}	b_{17}	b_{19}
$\frac{1}{6}$	$\frac{1}{30}$	$\frac{1}{42}$	$\frac{1}{30}$	$\frac{5}{66}$	$\frac{691}{2730}$	$\frac{7}{6}$	$\frac{3617}{510}$	$\frac{43867}{798}$	$\frac{174611}{330}$

b_{21}	b_{23}	b_{25}	b_{27}	b_{29}
$\frac{854513}{138}$	$\frac{236364091}{2730}$	$\frac{8553103}{6}$	$\frac{23749461029}{870}$	$\frac{8615841276005}{14322}$. *)

*) Quod si valorem numeri cujuscumque Bernoulliani, ab antecedentibus non pendentem, formula generali expressum velis, haberi observandum est expressionem sequentem *Laplacio* debitam:

$$b_{2p-1} = \frac{2p}{(2^{2p}-1)\,2^{2p-1}} \Big\{ \tfrac{1}{2} p^{2p-1} - (p-1)^{2p-1} \left\{1 + \tfrac{1}{2}\tfrac{2p}{1}\right\}$$

$$+ (p-2)^{2p-1} \left\{1 + \frac{2p}{1} + \tfrac{1}{2}.\frac{2p(2p-1)}{1.2}\right\} - (p-3)^{2p-1}$$

$$\left\{1 + \frac{2p}{1} + \frac{2p(2p-1)}{1.2} + \tfrac{1}{2}.\frac{2p(2p-1)(2p-2)}{1.2.3}\right\} + \&c. \ldots\ldots$$

$$\pm (p-m)^{2p-1} \left\{1 + \frac{2p}{1} + \frac{2p(2p-1)}{1.2} + \frac{2p(2p-1)(2p-2)}{1.2.3}\right.$$

$$+ \ldots \frac{2p(2p-1)(2p-2)\ldots(2p-(m-2))}{1.2.3\ldots(m-1)}$$

$$\left. + \tfrac{1}{2}.\frac{2p(2p-1)(2p-2)\ldots(2p-(m-1))}{1.2.3\ldots m}\right\} \ldots \Big\},$$

ubi + scilicet adhibendum ante terminum generalem est, si m numerus est par, — autem si m impar, notandum.

3:o Porro, si brevitatis gratia ponamus

$$\Sigma\Sigma p_x = \Sigma^2 p_x$$

$$\Sigma\Sigma\Sigma p_x = \Sigma^3 p_x$$

$$\Sigma\Sigma\Sigma\Sigma p_x = \Sigma^4 p_x$$

$$\cdots\cdots$$

sequentem habebimus formulam non minus notatu dignam

$$\begin{aligned}\Sigma p_x q_x = p_x\ \Sigma q_x \\ &- (p_{x+1} - p_x)(\Sigma^2 q_x + \Sigma q_x) \\ &+ (p_{x+2} - 2p_{x+1} + p_x)(\Sigma^3 q_x + 2\Sigma^2 q_x + \Sigma q_x) \\ &- (p_{x+3} - 3p_{x+2} + 3p_{x+1} - p_x)(\Sigma^4 q_x + 3\Sigma^3 q_x \\ &+ 3\Sigma^2 q_x + \Sigma q_x) \\ &+\end{aligned}$$

que est, omnes in casu quolibet particulari omittendos esse terminos, ubi $m > p$. Sic v. gr. si $p = 2$, habebitur

$$b_3 = \frac{4}{(2^4 - 1).2^3} \cdot \left\{ \tfrac{1}{2} \cdot 2^3 - 1^3 \cdot (1 + \tfrac{1}{2} \cdot 4) \right\} = \tfrac{1}{30}.$$

Perspectu autem facile est, allatam nuper formulam magni non esse usus ad numerorum de quibus agitur determinationem, cum, crescente ipsa p, prolixior omnino multo ejusdem fieret applicatio, ac si, per allatas supra formulas, antecedentium ope quivis determinaretur numerus.

$$+ (p_{x+4} - 4p_{x+3} + 6p_{x+2} - 4p_{x+1} + p_x)$$

$$(\Sigma^5 q_x + 4\Sigma^4 q_x + 6\Sigma^3 q_x + 4\Sigma^2 q_x + \Sigma q_x)$$

$$- (p_{x+5} - 5p_{x+4} + 10p_{x+3} - 10p_{x+2} + 5p_{x+1} - p_x)$$

$$(\Sigma^6 q_x + 5\Sigma^5 q_x + 10\Sigma^4 q_x + 10\Sigma^3 q_x + 5\Sigma^2 q_x$$

$$+ \Sigma q_x)$$

$$+ \&c. \quad \ldots\ldots\ldots\ldots \quad b)$$

quæ pro quacumque valet forma ipsarum p_x et q_x, cujusque lex manifesta est, coefficientibus numericis eadem scilicet progredientibus lege, ac in evolutione potestatum binomii.

4:o Tandemque ad formulas attendamus duas particulares usus, ut infra elucebit, in præsenti materie haud contemnendi, sequentes scilicet

$$\Sigma \frac{1}{x(x+1)(x+2)(x+3)\ldots(x+m)} = \frac{-1}{mx(x+1)(x+2)(x+3)\ldots(x+m-1)} \ldots c)$$

et

$$\Sigma a^x \operatorname{Sin}(b + cx) = \frac{a^x [a \operatorname{Sin}(b - c + cx) - \operatorname{Sin}(b + cx)]}{a^2 - 2a \operatorname{Cos} c + 1} \ldots d),$$

ubi m numerus quicumque est integer, ipsasque a, b et c ab x non pendere assumitur.

§. VI.

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His quidem formulis omnes fere nituntur quae hucusque nobis constant ad determinandam functionem Σv regulae paullo generaliores, modo notentur adhuc sequentia, quae applicationem earumdem illustrantia addere necessum est.

Posito in formula *a*)

$$v = x^m,$$

abibit ea in

$$\Sigma x^m = \frac{x^{m+1}}{m+1} - \tfrac{1}{2}\, x^m + \tfrac{1}{6} \cdot \frac{m}{1.2}\, x^{m-1} - \tfrac{1}{30} \cdot \frac{m\,(m-1)\,(m-2)}{1.2.3.4}\, x^{m-3}$$

$$+ \tfrac{1}{42} \cdot \frac{m\,(m-1)\,(m-2)\,..\,(m-4)}{1.2.3.4.5.6}\, x^{m-5} - \tfrac{1}{30} \cdot \frac{m(m-1)\,..\,(m-6)}{1.2\,..\,8}\, x^{m-7}$$

$$+ \frac{5}{66} \cdot \frac{m\,(m-1)\,..\,(m-8)}{1.2\,..\,10}\, x^{m-9} - \frac{691}{2730} \cdot \frac{m\,(m-1)\,..\,(m-10)}{1.2\,..\,12}\, x^{m-11}$$

$$+ \frac{7}{6} \cdot \frac{m\,(m-1)\,..\,(m-12)}{1.2\,..\,14}\, x^{m-13} - \frac{3617}{510} \cdot \frac{m\,(m-1)\,..\,(m-14)}{1.2\,..\,16}\, x^{m-15}$$

$$+ \frac{43867}{798} \cdot \frac{m(m-1)..(m-16)}{1.2\,..\,18}\, x^{m-17} - \frac{174611}{330} \cdot \frac{m(m-1)..(m-18)}{1.2\,..\,20}\, x^{m-19}$$

$$+ \frac{854513}{138} \cdot \frac{m\,(m-1)\,..\,(m-20)}{1.2\,..\,22}\, x^{m-21} - \frac{236364091}{2730} \cdot \frac{m\,(m-1)\,..\,(m-22)}{1.2\,..\,24}$$

$$x^{m-23}$$

$$x^{m-23} + \frac{8553103}{6} \cdot \frac{m\,(m-1)\,..\,(m-24)}{1\,.\,2\,..\,26}\, x^{m-25} - \frac{23749461029}{870}$$

$$\cdot \frac{m\,(m-1)\,..\,(m-26)}{1\,.\,2\,..\,28}\, x^{m-27} + \frac{8615841276005}{14322} \cdot \frac{m\,(m-1)\,..\,(m-28)}{1\,.\,2\,..\,30}$$

$$x^{m-29} - \&c., \quad \ldots\ldots\ldots\ldots \quad e)$$

quæ quidem series necessario abrumpitur formaque se offert finita quoties m numerus est positivus integer. Quandoquidem commodum sæpenumero est valores aliquos hujus formulæ particulares in promtu habere, sequentem hunc in finem tabellam attulisse alienum non erit:

$$\Sigma x^0 = x$$

$$\Sigma x^1 = \tfrac{1}{2} x^2 - \tfrac{1}{2} x$$

$$\Sigma x^2 = \tfrac{1}{3} x^3 - \tfrac{1}{2} x^2 + \tfrac{1}{6} x$$

$$\Sigma x^3 = \tfrac{1}{4} x^4 - \tfrac{1}{2} x^3 + \tfrac{1}{4} x^2$$

$$\Sigma x^4 = \tfrac{1}{5} x^5 - \tfrac{1}{2} x^4 + \tfrac{1}{3} x^3 - \tfrac{1}{30} x$$

$$\Sigma x^5 = \tfrac{1}{6} x^6 - \tfrac{1}{2} x^5 + \tfrac{5}{12} x^4 - \tfrac{1}{12} x^2$$

$$\Sigma x^6 = \tfrac{1}{7} x^7 - \tfrac{1}{2} x^6 + \tfrac{1}{2} x^5 - \tfrac{1}{6} x^3 + \tfrac{1}{42} x$$

$$\Sigma x^7 = \tfrac{1}{8} x^8 - \tfrac{1}{2} x^7 + \tfrac{7}{12} x^6 - \tfrac{7}{24} x^4 + \tfrac{1}{12} x^2$$

$$\Sigma x^8 = \tfrac{1}{9} x^9 - \tfrac{1}{2} x^8 + \tfrac{2}{3} x^7 - \tfrac{7}{15} x^5 + \tfrac{2}{9} x^3 - \tfrac{1}{30} x$$

Σx^9

$$\Sigma x^9 = \tfrac{1}{10} x^{10} - \tfrac{1}{2} x^9 + \tfrac{3}{4} x^8 - \tfrac{7}{10} x^6 + \tfrac{1}{2} x^4 - \tfrac{3}{20} x^2$$

$$\Sigma x^{10} = \tfrac{1}{11} x^{11} - \tfrac{1}{2} x^{10} + \tfrac{5}{6} x^9 - x^7 + x^5 - \tfrac{1}{2} x^3 + \tfrac{5}{66} x$$

&c. &c.

Posita

$$gx^p + hx^q + \text{\&c.}$$

functione quacumque rationali integra, innotescere per formulam *e*) in genere patet

$$\Sigma\,(gx^p + hx^q + \text{\&c.}),$$

cum habeatur scilicet per præcedentia

$$\Sigma\,(gx^p + hx^q + \text{\&c.}) = g\Sigma x^p + h\Sigma x^q + \text{\&c.}\ ^{*)}$$

Usus vero formulæ *b*) in eo præcipue cernitur, quod ejus ope in terminis finitis exhiberi semper possit

$$\Sigma p_x q_x,$$

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*) Observari tamen convenit, haberi functionem quamdam rationalem integram formæ particularis sæpeque satis obvenientem, istam scilicet

$$x\,(x + 1)\,(x + 2)\,.\,.\,.\,(x + m),$$

cujus tractatio nullis obnoxia sit ambagibus. Est scilicet, ut perspicitur facile,

$$\Sigma x\,(x+1)\,(x+2)\,.\,.\,(x+m) = \frac{(x-1)\,x\,(x+1)\,(x+2)\,.\,.\,(x+m)}{m+2}.$$

quoties p_x talis est functio, ut evanescant tandem coefficientes

$$p_x$$

$$p_{x+1} - p_x$$

$$p_{x+2} - 2p_{x+1} + p_x$$

$$p_{x+3} - 3p_{x+2} + 3p_{x+1} - p_x,$$

&c.

simulque q_x talis, ut in genere determinari queat

$$\Sigma^n q_x.$$

Probari vero facile potest, primae satisfacturam esse conditioni functionem quamlibet rationalem integram, i. e. formae

$$gx^p + hx^q + \&c.,$$

pro p_x acceptam *); nullique etiam obnoxiam esse diffi-

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*) Quod quidem eo fieri potest modo, ut posito

$$p_x = gx^p + hx^q + \&c.,$$

observetur haberi

$$p_{x+1} - p_x = g(x+1)^p + h(x+1)^q + \&c. - gx^p - hx^q - \&c.$$

difficultati determinationem generalem ipsius

$$\Sigma^n q_x;$$

assumtis

$$p_{x+1} - p_x = \left.\begin{matrix} g\,(p x^{p-1} + \frac{p(p-1)}{1.2} x^{p-2} + \&c.) \\ + h\,(q x^{q-1} + \frac{q(q-1)}{1.2} x^{q-2} + \&c.) \\ + \&c. \end{matrix}\right\}$$

$$= \left.\begin{matrix} g' x^{p-1} + g'_{,} x^{p-2} + g'_{,,} x^{p-3} + \&c. \\ + h' x^{q-1} + h'_{,} x^{q-2} + h'_{,,} x^{q-3} + \&c. \\ + \&c. \end{matrix}\right\}$$

$$= p'_x;$$

$$p_{x+2} - 2p_{x+1} + p_x = p'_{x+1} - p'_x$$

$$= \left.\begin{matrix} g'(x+1)^{p-1} + g_{,}'(x+1)^{p-2} + \&c. \\ - g' x^{p-1} - g_{,}' x^{p-2} - \&c. \\ + h'(x+1)^{q-1} + h_{,}'(x+1)^{q-2} + \&c. \\ - h' x^{q-1} - h_{,}' x^{q-2} - \&c. \\ + \&c. \end{matrix}\right\}$$

$$= \left.\begin{matrix} g'\,((p-1)\,x^{p-2} + \frac{(p-1)(p-2)}{1.2} x^{p-3} \\ + \&c.) + g_{,}'\,((p-2)\,x^{p-3} \\ + \frac{(p-2)(p-3)}{1.2} x^{p-4} + \&c.) + \&c. \end{matrix}\right\}$$